



Machon, T. (2020). The Godbillon–Vey invariant as a restricted Casimir of three-dimensional ideal fluids. *Journal of Physics A: Mathematical and Theoretical*, 53(23), [235701].
<https://doi.org/10.1088/1751-8121/ab8b35>

Publisher's PDF, also known as Version of record

License (if available):
CC BY

Link to published version (if available):
[10.1088/1751-8121/ab8b35](https://doi.org/10.1088/1751-8121/ab8b35)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the final published version of the article (version of record). It first appeared online via IOP Publishing at <https://iopscience.iop.org/article/10.1088/1751-8121/ab8b35/meta> . Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

PAPER • OPEN ACCESS

The Godbillon–Vey invariant as a restricted Casimir of three-dimensional ideal fluids^{*}

To cite this article: Thomas Machon 2020 *J. Phys. A: Math. Theor.* **53** 235701

View the [article online](#) for updates and enhancements.



IOP | ebooks™

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

The Godbillon–Vey invariant as a restricted Casimir of three-dimensional ideal fluids*

Thomas Machon 

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, United Kingdom

Received 5 January 2020, revised 16 March 2020

Accepted for publication 20 April 2020

Published 18 May 2020



CrossMark

Abstract

We show the Godbillon–Vey invariant arises as a ‘restricted Casimir’ invariant for three-dimensional ideal fluids associated to a foliation. We compare to a finite-dimensional system, the rattleback, where analogous phenomena occur.

Keywords: topological fluid dynamics, Hamiltonian systems, Casimir invariants

1. Introduction

The topological aspect of ideal fluids has its origins in the transport of vorticity. A consequence is the conservation of helicity; in the Hamiltonian formulation of ideal fluids as an infinite-dimensional Lie–Poisson system, helicity appears as a Casimir invariant, a degeneracy in the Lie–Poisson bracket [1]. The goal of this paper is to show that a higher-order Casimir, the Godbillon–Vey invariant, can be defined for ideal fluids in a certain subregion of phase space.

The state of an ideal fluid on a homology three-sphere is specified by the vorticity, a divergence-free vector field. A Casimir in an ideal fluid is invariant under all volume-preserving diffeomorphisms of the domain, so can be said to measure a topological property of the vorticity, for example helicity measures the average linking of vortex lines [2–4]. Any Casimir that can be written as a regular integral invariant of all vorticity fields is equivalent to helicity [5–7]; accordingly, higher-order regular integral invariants can only be defined for special subclasses of vorticity fields. Here we study the Godbillon–Vey invariant (GV) which can be associated to a vorticity field tangent to a codimension-1 foliation [8–11]. GV originates in the theory of

* I am extremely grateful to both PJ Morrison and D Peralta-Salas for hugely enlightening discussions. I would also like to acknowledge many useful conversations with J H Hannay.



Original content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

foliations [12, 13]; in ideal fluids it measures topological helical compression of vortex lines [8]. We show how GV fits naturally into the Lie–Poisson Hamiltonian formulation of ideal fluids [1] as a ‘restricted Casimir’ invariant. In particular, we consider a set S of ideal fluids where the Lie–Poisson bracket has an additional degeneracy associated to the Lie subalgebra of volume-preserving vector fields tangent to a foliation, which may vary within S . On S we construct a modified Lie–Poisson type bracket, in terms of which the Godbillon–Vey invariant appears as a Casimir.

The configurations for which GV is defined always have vanishing helicity and in this sense GV is hierarchical, in a manner analogous to that suggested by Arnold and Khesin [14]. Recent work [15, 16] has studied similar hierarchical structures in Hamiltonian systems, where a singular region in phase space with a Poisson operator of decreased rank can itself be considered as a Poisson submanifold, on which new Casimir invariants appear. What we describe can be considered an example of this phenomenon.

A finite-dimensional example is found in the Lie–Poisson formulation of the ‘rattleback’ spinning top [17], where corresponding phenomena occur: there is a submanifold of phase space where the Poisson operator has an additional degeneracy associated to a Lie subalgebra; on this submanifold the primary Casimir vanishes and a new restricted Casimir appears. In the finite-dimensional rattleback case, perturbation of the system around the singular manifold leads to interesting dynamical properties [17]. Our own analysis of the Godbillon–Vey invariant elsewhere also suggests a strong connection to dynamics; GV provides a global and local obstruction to steady flow and can be used to estimate the rate of change of vorticity [8]. With that in mind, we suggest that flows with $GV \neq 0$ (or perturbations thereof) may prove particularly interesting from a dynamical perspective.

2. Lie–Poisson systems

See e.g. [18] for a description. Let \mathfrak{g} be a Lie algebra associated to a group G , with \mathfrak{g}^* its dual. Given an element $\alpha \in \mathfrak{g}^*$ and two elements $U, V \in \mathfrak{g}$ we form the bracket

$$\langle \alpha, [U, V] \rangle, \quad (1)$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is the natural pairing between the Lie algebra and its dual, and $[\cdot, \cdot]$ is the Lie bracket of \mathfrak{g} . This is then used to define the Lie–Poisson bracket

$$\{F, G\}_{\pm} = \pm \left\langle \alpha, \left[\frac{\delta F}{\delta \alpha}, \frac{\delta G}{\delta \alpha} \right] \right\rangle, \quad (2)$$

where the (functional) derivative $\delta F / \delta \alpha$ is identified with an element of \mathfrak{g} by the relation

$$\frac{d}{d\epsilon} F(\alpha + \epsilon \delta \alpha)|_{\epsilon=0} = \left\langle \delta \alpha, \frac{\delta F}{\delta \alpha} \right\rangle. \quad (3)$$

The sign in (2) depends on whether we consider right-invariant or left-invariant function(al)s on \mathfrak{g}^* with respect to the coadjoint representation of G , but is irrelevant for our purposes. Coupled with a Hamiltonian function on \mathfrak{g}^* , this specifies the system. The noncanonical nature of the Lie–Poisson bracket allows for the existence of Casimir invariants, C , given by the property $\{F, C\} = 0$ for any function F . We define the coadjoint bracket $[\cdot, \cdot]^{\dagger} : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ as

$$\langle [U, \alpha]^{\dagger}, V \rangle = \langle \alpha, [U, V] \rangle. \quad (4)$$

This allows us to give the condition for C to be a Casimir as

$$\left[\frac{\delta C(\alpha)}{\delta \alpha}, \alpha \right]^\dagger = 0. \quad (5)$$

In this paper we will be interested in sets of elements $\alpha \in S \subset \mathfrak{g}^*$ where there is a non-generic degeneracy associated to a subalgebra $\mathfrak{h}_\alpha \subset \mathfrak{g}$, such that

$$\langle \alpha, U \rangle = 0, \quad (6)$$

for $U \in \mathfrak{h}_\alpha$. For a given $\alpha \in \mathfrak{g}^*$, let $\beta = \text{ad}_g^* \alpha$, $g \in G$. Then β is orthogonal to the subalgebra $\mathfrak{h}_\beta = \text{ad}_g \mathfrak{h}_\alpha$, so that S will, in general, be a set of coadjoint orbits in \mathfrak{g}^* . The precise specification of admissible subalgebras and the subset S in a general formulation is left intentionally vague.

3. Finite dimensional example: the rattleback

An idealised description of the chiral dynamics of a rattleback spinning top [17] can be formulated as a Lie–Poisson system based on the three-dimensional Lie algebra with Bianchi classification $\text{VI}_{h < -1}$, spanned by three elements, P, R, S with Lie bracket

$$[P, R] = 0, \quad [S, P] = hP, \quad [S, R] = R. \quad (7)$$

Physically P, R , and S are associated to pitching, rolling and spinning motions respectively, and h is a geometric parameter related to the aspect ratio of the top. The dynamical variable is an element of the dual space \mathfrak{g}^* which we write as a lowercase triple (p, r, s) , in terms of which the dynamics are [17, 19]

$$\frac{d}{dt} \begin{pmatrix} p \\ r \\ s \end{pmatrix} = \begin{pmatrix} -hps \\ -rs \\ r^2 + hp^2 \end{pmatrix}. \quad (8)$$

The Hamiltonian of this system is given by $H = (p^2 + r^2 + s^2)/2$. At a generic point in \mathfrak{g}^* the Lie–Poisson bracket has a one-dimensional kernel, associated to the Casimir

$$C = pr^{-h}, \quad (9)$$

which one can check is conserved by the dynamics (8).

There is a two-dimensional Abelian subalgebra $\mathfrak{h} \subset \text{VI}_{h < -1}$, spanned by P, R . The set of points $L \subset \mathfrak{g}^*$ orthogonal to \mathfrak{h} is the singular line $(0, 0, s)$, so that on L the Casimir $C = 0$. On L the Lie–Poisson bracket is trivial, so the dynamics are trivial (one can see this by setting $p = r = 0$ in (8)). It follows that s is a constant of the motion on L only. Finally, note that L can be thought of as a one-dimensional Poisson manifold with trivial Poisson bracket, and with respect to this bracket s is a Casimir invariant (as is any function of s), so that s is a restricted Casimir invariant of the rattleback system. Physically it corresponds to simple spinning motion of the top.

4. The Godbillon–Vey invariant as a restricted Casimir in three-dimensional ideal fluids

Now we see how the same pattern of phenomena is found in three-dimensional ideal fluids on a manifold M . We assume throughout that M is a homology three-sphere (one can take $M = S^3$).

4.1. Ideal Hydrodynamics and helicity

In the Lie–Poisson formulation of ideal fluids [1, 14], \mathfrak{g} is the Lie algebra of volume-preserving vector fields on M with respect to a volume form μ , so that $\mathcal{L}_U \mu = 0$ for $U \in \mathfrak{g}$ and the two-form $\iota_U \mu$ is closed. The dynamical variable is given by an element of the dual space \mathfrak{g}^* , the smooth part of which can be identified as $\Omega^1(M)/d\Omega^0(M)$, the space of differential one-forms modulo exact forms, and each element is given by a coset $[\alpha]$, with specific representative α . We will suppress the coset notation $[\cdot]$. The pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is given by

$$\langle \alpha, U \rangle = \int_M (\iota_U \alpha) \mu, \quad (10)$$

which does not depend on the representative one-form α . In this case the Lie–Poisson bracket takes the form

$$\langle \alpha, [U, V] \rangle = \int_M \alpha \wedge \iota_{[U, V]} \mu. \quad (11)$$

The coadjoint bracket is given as

$$[U, \alpha]^\dagger = -\iota_U d\alpha = -\iota_U \iota_W \mu, \quad (12)$$

where the vorticity field $W \in \mathfrak{g}$ is given by $d\alpha = \iota_W \mu$. Helicity is defined as

$$\mathcal{H} = \int_M \alpha \wedge d\alpha = \langle \alpha, W \rangle. \quad (13)$$

A short calculation gives

$$\frac{d}{d\epsilon} \mathcal{H}(\alpha + \epsilon \delta \alpha) \Big|_{\epsilon=0} = \int_M \delta \alpha \wedge 2d\alpha, \quad (14)$$

so $\delta \mathcal{H} / \delta \alpha = 2W$ and hence

$$\left[\frac{\delta \mathcal{H}}{\delta \alpha}, \alpha \right]^\dagger = 0, \quad (15)$$

so that \mathcal{H} is a Casimir.

4.2. Codimension-1 foliations

A codimension-1 foliation \mathcal{F} of M is a decomposition of M into two-dimensional leaves. Locally a foliation in a small ball $B^3 \subset M$ is a decomposition $B^3 \cong D^2 \times [0, 1]$, where the disk, D^2 , describes the two dimensional leaves, and $[0, 1]$ is the transverse direction. Globally the properties of \mathcal{F} can be extremely complicated.

Any \mathcal{F} can be defined by a non-vanishing one-form β . At any point $p \in M$, the two-dimensional space of vectors orthogonal to β (i.e. vectors satisfying $\iota_X \beta = 0$ at p) defines the tangent space to the leaf of \mathcal{F} passing through p . The requirement that the tangent spaces at each point p stitch together to form leaves of \mathcal{F} is given by Frobenius' integrability condition

$$\beta \wedge d\beta = 0, \quad (16)$$

this also holds for any one-form $f\beta$ with f a non-zero function, which defines the same foliation. Frobenius' theorem also states that the Lie bracket of two volume-preserving vector fields X and Y , tangent to \mathcal{F} is also tangent to \mathcal{F} so that the set of volume-preserving vector fields tangent to the foliation is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

4.3. The Godbillon–Vey invariant

For a codimension-1 foliation \mathcal{F} on a closed manifold M , the Godbillon–Vey class [12, 13] is an element $\text{GV} \in H^3(M; \mathbb{R})$, if M is a closed three-manifold and $H^3(M; \mathbb{R}) = \mathbb{R}$ and $\text{GV} \in \mathbb{R}$ is a diffeomorphism invariant of the foliation. Let β be a defining one-form for \mathcal{F} , then the integrability condition $\beta \wedge d\beta = 0$ implies there is a one-form η such that

$$d\beta = \beta \wedge \eta. \quad (17)$$

The three-form $\eta \wedge d\eta$ is closed and GV is defined as

$$\text{GV} = \int_M \eta \wedge d\eta, \quad (18)$$

β is only defined up to multiplication by a non-zero function, and η is only defined up to addition of a multiple of β , but under these transformations $\eta \wedge d\eta$ changes by an exact three-form, so GV is well-defined. By construction GV is a diffeomorphism invariant of \mathcal{F} . Finally, note that by differentiating (17) we get

$$0 = \beta \wedge d\eta, \quad (19)$$

hence $d\eta = \beta \wedge \gamma$, for some one-form γ . GV can be thought of as helical compression of vortex lines [8], with η the local direction of vorticity compression.

4.4. Foliations and \mathfrak{g}^*

We now consider a particular subset of the fluid configuration space $S \subset \mathfrak{g}^*$, so that for each $\alpha \in S$ there is a codimension-1 foliation \mathcal{F}_α of the fluid domain M such that

$$\langle \alpha, X \rangle = 0, \quad \forall X \in \mathfrak{h}_\alpha, \quad (20)$$

where $\mathfrak{h}_\alpha \subset \mathfrak{g}$ is the subalgebra of volume-preserving vector fields tangent to \mathcal{F}_α .

The goal of this section is to show that (20) implies two things. Firstly, that the helicity vanishes and the vorticity is tangent to \mathcal{F}_α . Secondly, that if α satisfies (20) then it must be a defining form for the foliation, up to multiplication by a function. In the physical case with M a subset of \mathbb{R}^3 with the Euclidean metric, α is a gauge transformation of the velocity field, and (20) implies that α is the normal vector field to a foliation of M , with the vorticity field tangent to the leaves.

First we show (20) implies the helicity vanishes. We define a subset of \mathfrak{h}_α by considering the closed two-forms $d(h\beta_\alpha)$, where β_α is a defining one-form for \mathcal{F}_α and h is a function. Then consider the vector field Y defined by $\iota_Y \mu = d(h\beta_\alpha)$. Since $\iota_Y \mu$ is closed, Y is volume preserving, and $(\iota_Y \beta_\alpha) \mu = \beta_\alpha \wedge d(h\beta_\alpha) = 0$, implying $\iota_Y \beta_\alpha = 0$ and so $Y \in \mathfrak{h}_\alpha$.

Now by the requirement (20)

$$0 = \langle \alpha, Y \rangle = \int_M \alpha \wedge d(h\beta_\alpha) = \int_M h\beta_\alpha \wedge d\alpha. \quad (21)$$

As h is arbitrary this implies $\beta_\alpha \wedge d\alpha = 0$. Recall the vorticity field W defined by $d\alpha = \iota_W \mu$, then $\beta_\alpha \wedge d\alpha = (\iota_W \beta_\alpha) \mu = 0$ implies the vorticity is tangent to \mathcal{F}_α , so that

$$W \in \mathfrak{h}_\alpha. \quad (22)$$

Combining (20) and (22) we find the helicity vanishes,

$$\mathcal{H} = \langle \alpha, W \rangle = 0, \quad \forall \alpha \in S. \quad (23)$$

(20) also implies that α is related to the defining form of the foliation. Indeed, the condition $\beta_\alpha \wedge d\alpha = 0$ states that α is closed when restricted to the leaves of \mathcal{F}_α , so that α defines a class $[\alpha]_\mathcal{F}$ in the leafwise cohomology group $H^1(\mathcal{F}_\alpha)$. Representative one-forms of a given class in $H^1(\mathcal{F}_\alpha)$ are elements of the coset $\Omega^1(M)/\mathcal{I}^1(\mathcal{F}_\alpha)$, where $\mathcal{I}^1(\mathcal{F}_\alpha)$ is the set of one-forms $f\beta_\alpha$, f an arbitrary function. Any element of $H^1(\mathcal{F}_\alpha)$ is thus written as $\rho + f\beta_\alpha$, where ρ is closed when restricted to \mathcal{F}_α . In fact

$$[\alpha]_\mathcal{F} = 0 \in H^1(\mathcal{F}_\alpha). \quad (24)$$

To show this, let γ be any closed loop tangent to \mathcal{F}_α . The circulation integral

$$I_\gamma = \int_\gamma \alpha \quad (25)$$

is invariant under leafwise homotopies of γ , since $\beta_\alpha \wedge d\alpha = 0$. Now consider a family of smooth vector fields $G_\lambda \in \mathfrak{g}$ with support in a tubular neighbourhood of γ of diameter $\sim \lambda$, tending to the singular vector field with support γ and constant flux ϕ as $\lambda \rightarrow 0$, so that $\int_D \iota_{G_\lambda} \mu \rightarrow \phi$ as $\lambda \rightarrow 0$, where D is a disk pierced by γ . Then $\langle \alpha, G_\lambda \rangle \rightarrow \phi I_g$ as $\lambda \rightarrow 0$. But $|\langle \alpha, G_\lambda \rangle| < \lambda C$ for some constant C , so $I_g = 0$. As γ was arbitrary, $[\alpha]_\mathcal{F} = 0 \in H^1(\mathcal{F}_\alpha)$. This implies any representative of the coset $[\alpha] \in \Omega^1(M)/d\Omega^0(M)$ can be written as $f\beta_\alpha + dg$ for functions f, g , or

$$[\alpha] \in \mathcal{S} \Rightarrow \alpha = f\beta_\alpha + dg. \quad (26)$$

There is then a canonical representative form $\alpha_c = f\beta_\alpha$, which we write as α in subsequent sections. With this choice of representative the helicity density vanishes, $\alpha_c \wedge \iota_W \mu = 0$.

4.5. $\Omega^1_t(M)$ and the functional derivative

From our previous results we may write $\alpha = f\beta_\alpha$, for simplicity we will henceforth assume that $f \neq 0$ so that the fluid state α is a defining one-form for the foliation \mathcal{F}_α . With this canonical choice, the subset of $\mathcal{S} \subset \mathfrak{g}^*$ we are considering can be identified with the space of nonvanishing integrable one-forms on M , which we write as $\Omega^1_t(M)$ (this space is not connected, we consider a single arbitrary connected component below). We note that $\Omega^1_t(M)$ is no longer a vector space.

We would like to define functional derivatives on $\Omega^1_t(M)$. For a one-parameter family $\alpha_t \in \Omega^1_t(M)$, $t \geq 0$ with time derivative $\dot{\alpha}_t$, we write $\dot{\alpha} = \dot{\alpha}_0$ and $\alpha = \alpha_0$. For a functional F on $\Omega^1_t(M)$ the functional derivative is defined as

$$\frac{d}{dt} F(\alpha_t)|_{t=0} = \left\langle \dot{\alpha}, \frac{\delta F}{\delta \alpha} \right\rangle, \quad (27)$$

and we identify $\delta F/\delta \alpha$ with a vector field as in section 4.1. Because we have a canonical choice of α , we no longer require invariance under gauge transformations $\alpha \rightarrow \alpha + df$ and so are not restricted to volume-preserving vector fields. We suppose instead $\delta F/\delta \alpha \in \mathfrak{X}(M)/\Xi_\alpha$ where $\mathfrak{X}(M)$ is the space of smooth vector fields on M and $\Xi_\alpha \subset \mathfrak{X}(M)$ is an α -dependent subset satisfying $\langle \dot{\alpha}, U \rangle = 0$ for $U \in \Xi_\alpha$.

Our characterisation of Ξ_α below is not complete, but is sufficient for our purposes. First, we will show that it is non-empty. As α_t is integrable we have

$$\alpha_t \wedge d\alpha_t = 0. \quad (28)$$

In particular this gives

$$0 = \frac{d}{dt} \left(\int_M f \alpha_t \wedge d\alpha_t \right) \Big|_{t=0} = \int_M \dot{\alpha} \wedge (f d\alpha + d(f\alpha)) \quad (29)$$

for any function f , so that fields V satisfying

$$\iota_V \mu = f d\alpha + d(f\alpha) \quad (30)$$

are elements of Ξ_α . Now we give two properties of general elements of Ξ_α .

Firstly we note that any field in Ξ_α must be tangent to \mathcal{F}_α , or

$$U \in \Xi_\alpha \Rightarrow \iota_U \alpha = 0. \quad (31)$$

We can choose $\alpha_t = \exp(gt)\alpha$, so that $\dot{\alpha} = g\alpha$ for an arbitrary function g . Now suppose U is not tangent to \mathcal{F}_α , then by an appropriate choice of g we can force $\langle g\alpha, U \rangle \neq 0$, so $U \notin \Xi_\alpha$.

Secondly we note that any element V of Ξ_α must satisfy $d(\iota_V \mu) = \eta \wedge (\iota_V \mu)$, where η is a one-form defined by the relation $d\alpha = \alpha \wedge \eta$. We can choose α_t to be generated by a family of diffeomorphisms, so that $\dot{\alpha} = \mathcal{L}_U \alpha$ for $U \in \mathfrak{X}(M)$. Suppose $V \in \Xi_\alpha$, then (31) implies that we may write $\iota_V \mu = \nu = \alpha \wedge \sigma$ and we require

$$0 = \int_M \mathcal{L}_U \alpha \wedge \alpha \wedge \sigma = \int_M (\iota_U \alpha) (d\alpha \wedge \sigma + d(\alpha \wedge \sigma)), \quad (32)$$

and since $\iota_U \alpha$ is arbitrary we find $d\alpha \wedge \sigma + d(\alpha \wedge \sigma) = 0$, or

$$V \in \Xi_\alpha \Rightarrow d(\iota_V \mu) = \eta \wedge \iota_V \mu. \quad (33)$$

Any element of Ξ_α must then be tangent to \mathcal{F}_α and satisfy (33). This is not a complete characterisation, there are vector fields satisfying (31) and (33) which are not elements of Ξ_α . This is demonstrated by example in section 4.7. We speculate that vector fields of the form (30) fully characterise Ξ_α .

4.6. The Poisson bracket on $\Omega_I^1(M)$

We define the Poisson bracket on $\Omega_I^1(M)$ which continues to take the standard form

$$\{F, G\}_I = \left\langle \alpha, \left[\frac{\delta F}{\delta \alpha}, \frac{\delta G}{\delta \alpha} \right] \right\rangle, \quad (34)$$

where now $\alpha \in \Omega_I^1(M)$ and the functional derivatives are cosets in $\mathfrak{X}(M)/\Xi_\alpha$. The bracket must not depend on the choice of representative vector field in $\mathfrak{X}(M)/\Xi_\alpha$ for each functional derivative. Consider a vector field A on M such that $\iota_A \alpha = 0$ and $d(\iota_A \mu) = \eta \wedge \iota_A \mu$, (properties (31) and (33)). From the previous section we know all elements of Ξ_α satisfy these conditions. Then

$$\langle \alpha, [A, V] \rangle = 0, \quad (35)$$

where $V \in \mathfrak{X}(M)$. We compute

$$\langle \alpha, [A, V] \rangle = \int_M \alpha \wedge \iota_{[A, V]} \mu = - \int_M \alpha \wedge \mathcal{L}_V \iota_A \mu. \quad (36)$$

Now $\iota_A \mu = \alpha \wedge \sigma$ for some one-form σ . Then we have

$$\langle \alpha, [A, V] \rangle = \int_M \alpha \wedge \sigma \wedge \mathcal{L}_V \alpha = - \int_M \iota_V \alpha (2d\alpha \wedge \sigma - \alpha \wedge d\sigma) = 0, \quad (37)$$

it follows that the bracket $\{F, G\}_I$ does not depend on the choice of representative vector field for the functional derivatives and becomes a Poisson bracket on $\Omega_I^1(M)$. Finally, we note that if F is the restriction to $\Omega_I^1(M)$ of a functional on \mathfrak{g}^* , then its functional derivative is still an element of \mathfrak{g} , all elements of which are representative vector fields of a coset in $\mathfrak{X}(M)/\Xi_\alpha$, and the bracket (34) reproduces the Lie–Poisson bracket of the original ideal fluid formulation. In particular, we can recover Euler’s equations by choosing the appropriate Hamiltonian.

4.7. The Godbillon–Vey invariant as a restricted Casimir

Our goal is to show that the Godbillon–Vey invariant is a Casimir with respect to the Poisson bracket $\{, \}_I$ defined above. Suppose we have a one-form $\alpha \in \Omega_I^1(M)$, and consider the variation of GV,

$$\frac{d}{dt}GV|_{t=0} = 2 \int_M \dot{\eta} \wedge d\eta = 2 \int_M \dot{\eta} \wedge \alpha \wedge \gamma. \quad (38)$$

Where we use the fact that $d\eta = \alpha \wedge \gamma$. Using $d\alpha = \alpha \wedge \eta$, we find $d\dot{\alpha} = \dot{\alpha} \wedge \eta + \alpha \wedge \dot{\eta}$, so that

$$\frac{d}{dt}GV|_{t=0} = 2 \int_M (\dot{\alpha} \wedge \eta - d\dot{\alpha}) \wedge \gamma = \int_M \dot{\alpha} \wedge 2(\eta \wedge \gamma - d\gamma). \quad (39)$$

We then arrive at the relation

$$\frac{\delta GV}{\delta \alpha} = X, \quad \iota_X \mu = 2(\eta \wedge \gamma - d\gamma) = \chi \quad (40)$$

Now we consider the two-form $\chi = 2(\eta \wedge \gamma - d\gamma)$, there is a freedom in χ arising from freedom in η and γ . We may make the transformations

$$\eta \rightarrow \eta + f\alpha, \quad \gamma \rightarrow \gamma + f\eta - df + g\alpha, \quad (41)$$

for functions f, g . Under these transformations one finds

$$\chi \rightarrow \chi - 2(gd\alpha + d(g\alpha)), \quad (42)$$

The two-form $2(gd\alpha + d(g\alpha))$ defines a vector field in Ξ_α , as per (29), which does not affect the value of dGV/dt . Now observe that χ satisfies $\alpha \wedge \chi = 0$ and $d\chi = \eta \wedge \chi$. So we find X is tangent to \mathcal{F}_α and satisfies (33). Using (35) we therefore conclude

$$\{F, GV\}_I = 0 \quad (43)$$

for any functional F on $\Omega_I^1(M)$ so that GV is a restricted Casimir of three-dimensional ideal fluids.

ORCID iDs

Thomas Machon  <https://orcid.org/0000-0001-8247-1030>

References

- [1] Morrison P J 1998 Hamiltonian description of the ideal fluid *Rev. Mod. Phys.* **70** 467–521
- [2] Moffatt H K 1969 The degree of knottedness of tangled vortex lines *J. Fluid Mech.* **35** 117–29
- [3] Arnold V I 1986 The asymptotic Hopf invariant and its applications *Sel. Math. Sov.* **5** 327–45
- [4] Moffatt H K and Ricca R L 1992 Helicity and the Călugăreanu invariant *Proc. R. Soc. Lond. A* **439** 411–29
- [5] Enciso A, Peralta-Salas D and Torres de Lizaur F 2016 Helicity is the only integral invariant of volume-preserving transformations *Proc. Natl Acad. Sci. USA* **113** 2035–40
- [6] Kudryavtseva E A 2014 Conjugation invariants on the group of area-preserving diffeomorphisms of the disk *Math. Notes* **95** 877–80
- [7] Kudryavtseva E A 2016 Helicity is the only invariant of incompressible flows whose derivative is continuous in C^1 -topology *Math. Notes* **99** 626–30
- [8] Machon T 2020 The Godbillon–Vey invariant as topological vorticity compression and obstruction to steady flow in ideal fluids (arXiv:2002.09992)
- [9] Webb G M, Dasgupta B, McKenzie J F, Hu Q and Zank G P 2014 Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics I: Lie dragging approach *J. Phys. A: Math. Theor.* **47** 095501
- [10] Webb G M, Prasad A, Anco S C and Hu Q 2019 Godbillon–Vey helicity and magnetic helicity in magnetohydrodynamics *J. Plasma Phys.* **85** 775850502
- [11] Tur A V and Yanovsky V V 1993 Invariants in dissipationless hydrodynamic media *J. Fluid Mech.* **248** 67–106
- [12] Godbillon C and Vey J 1971 Un invariant des feuilletages de codimension 1 *C. R. Acad. Sci. Ser. A* **273** 92–5
- [13] Candel A and Conlon L 1999 *Foliations I* (Providence, RI: AMS)
- [14] Arnold V I and Khesin B 1999 *Topological Methods in Hydrodynamics* (New York: Springer)
- [15] Yoshida Z and Morrison P J 2014 A hierarchy of noncanonical Hamiltonian system: circulation laws in an extended phase space *Fluid Dyn. Res.* **46** 031412
- [16] Yoshida Z and Morrison P J 2016 Hierarchical structure of noncanonical Hamiltonian systems *Phys. Scripta* **91** 024001
- [17] Yoshida Z, Tokieda T and Morrison P J 2017 Rattleback: a model of how geometric singularities induce dynamic chirality *Phys. Lett. A* **381** 2772–7
- [18] Thiffeault J-L and Morrison P J 2000 Classification and Casimir invariants of Lie–Poisson brackets *Physica D* **136** 205–44
- [19] Moffatt H K and Tokieda T 2008 Celt reversals: a prototype of chiral dynamics *Proc. R. Soc. Edinb.* **138** 361–8